Advanced Core in Algorithm Design # 1
算法設計要論 第1回

Yasushi Kawase
河瀬 康志

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Outline

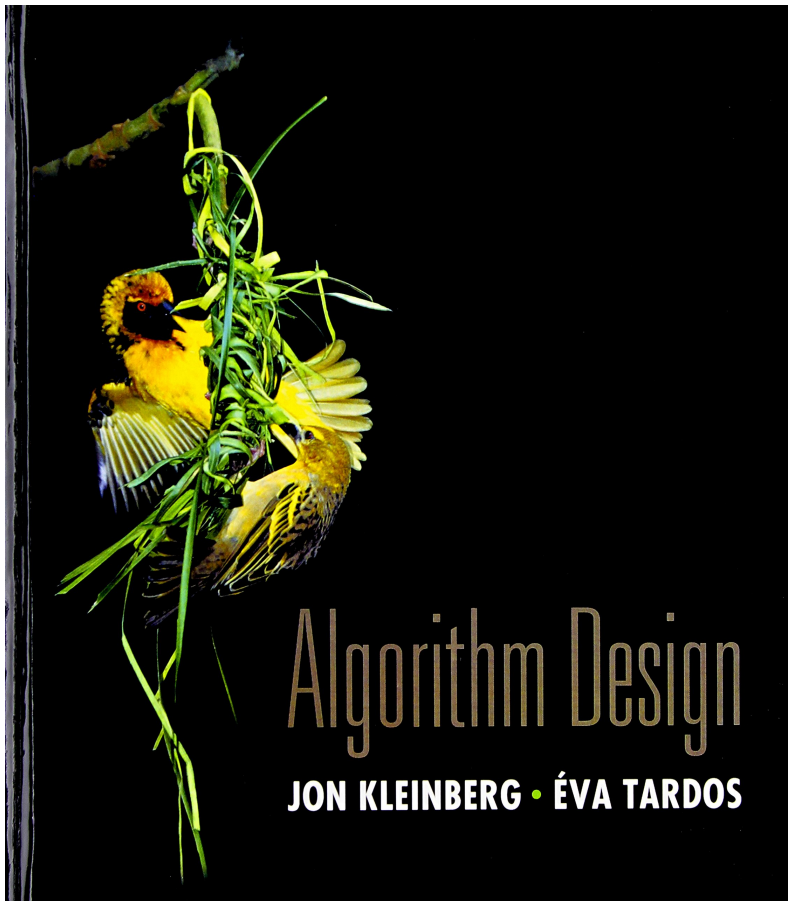
Grading

- Midterm report (50%)
- Final report (50%)

日本語 or English

Textbook

Jon Kleinberg and Éva Tardos: Algorithm Design



Schedule

Lec. #	Date	Topics
1	10/4	Introduction, Stable matching
2	10/11	Basics of Algorithm Analysis, Greedy Algorithms (1/2)
3	10/18	Greedy Algorithms (2/2)
4	10/25	Divide and Conquer (1/2)
5	11/1	Divide and Conquer (2/2)
6	11/8	Dynamic Programming (1/2)
7	11/15	Dynamic Programming (2/2)
—	11/22	Thursday Classes
8	11/29	Network Flow (1/2)
9	12/6	Network Flow (2/2)
10	12/13	NP and Computational Intractability
11	12/20	Approximation Algorithms (1/2)
12	12/27	Approximation Algorithms (2/2)
13	1/10	Randomized Algorithms

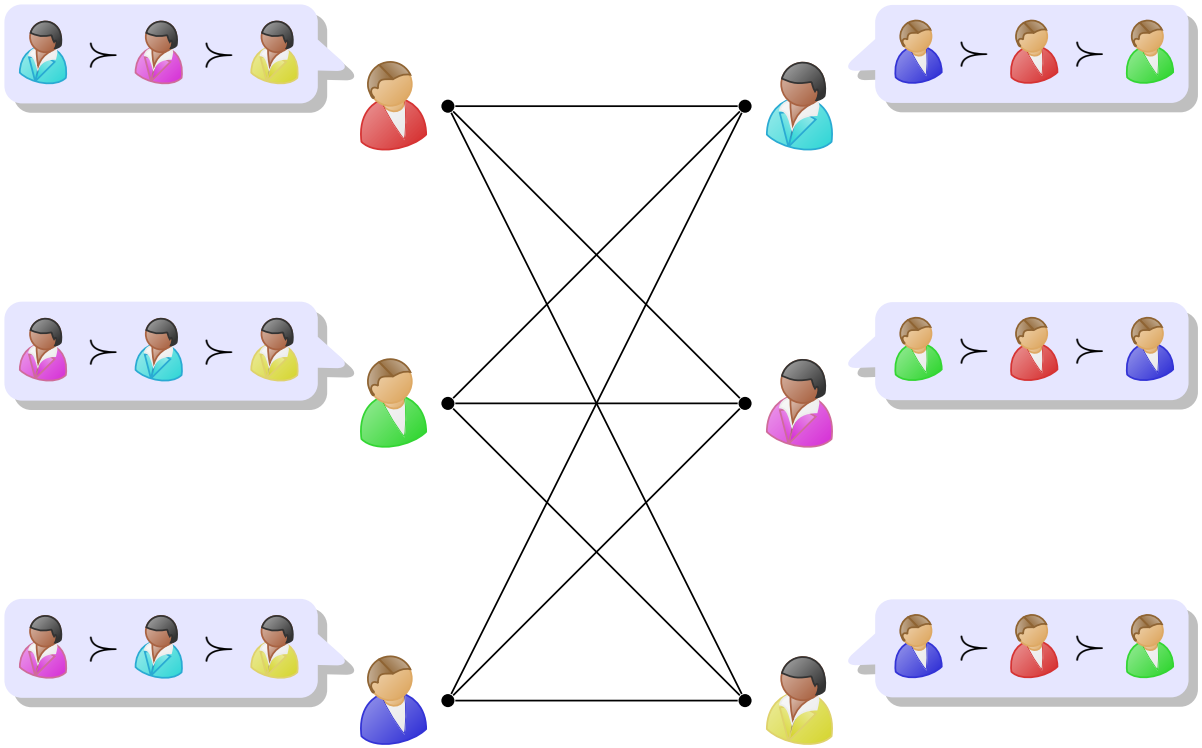
What is an algorithm?

- a set of rules to solve a particular problem
- stems from al-Khwarizmi
  - c. 780– c. 850
  - a Persian mathematician, astronomer, and geographer
  - In the 12th century, Latin translations of his work on the Indian numerals introduced the decimal number system to the Western world

http://www.muslimheritage.com/topics/default.cfm?ArticleID=631

Outline

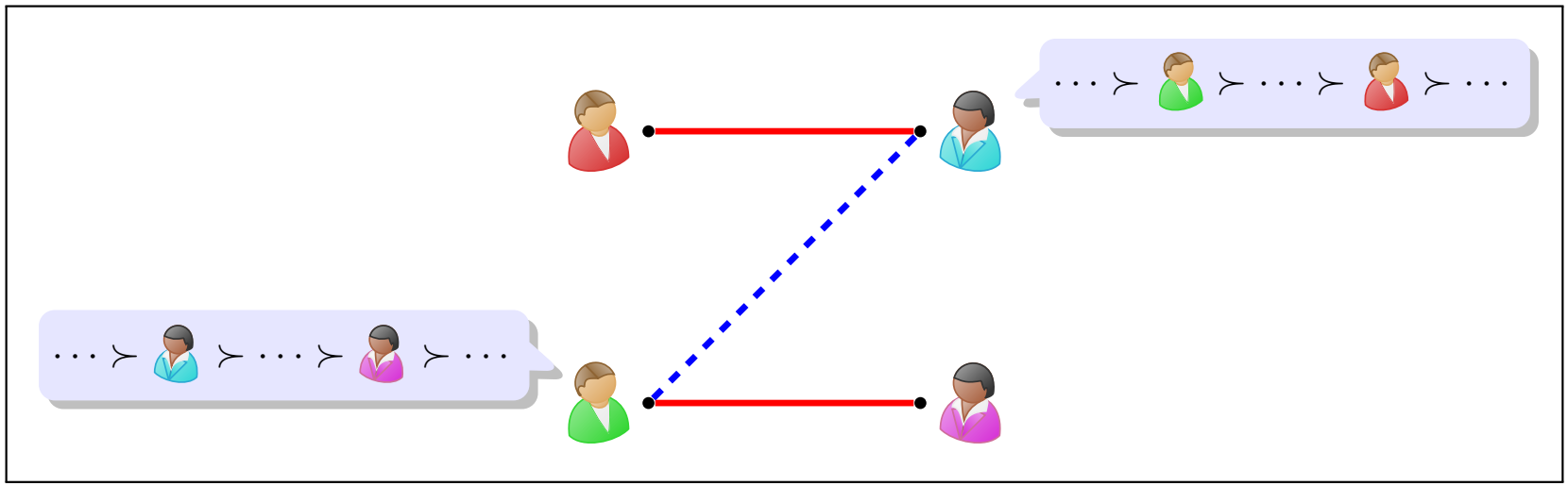
Model

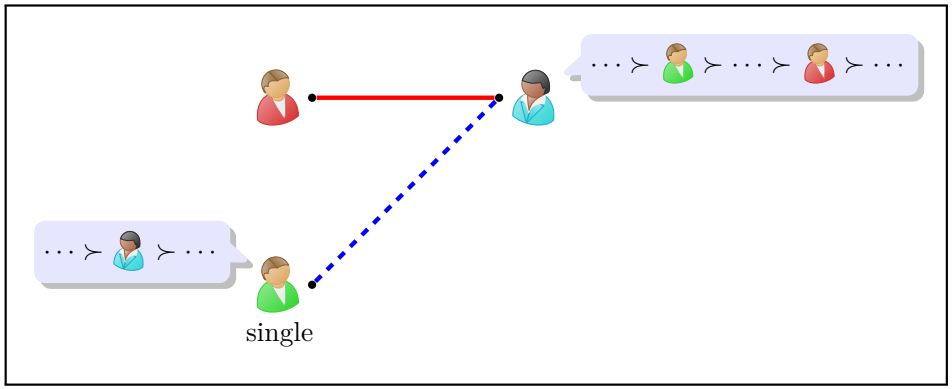


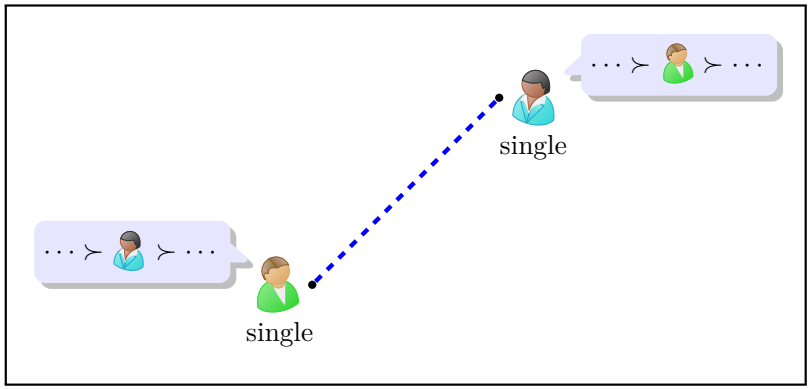
- n men and n women
- each man (woman) ranks all the woman (man)
- goal: to find a "stable" matching

Blocking Pair

a matching is unstable if blocking pairs (pairs like the following) exist

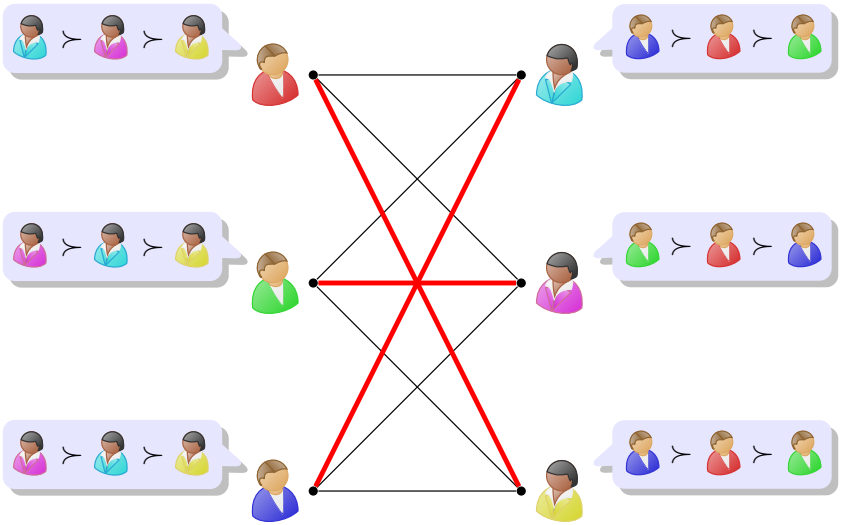




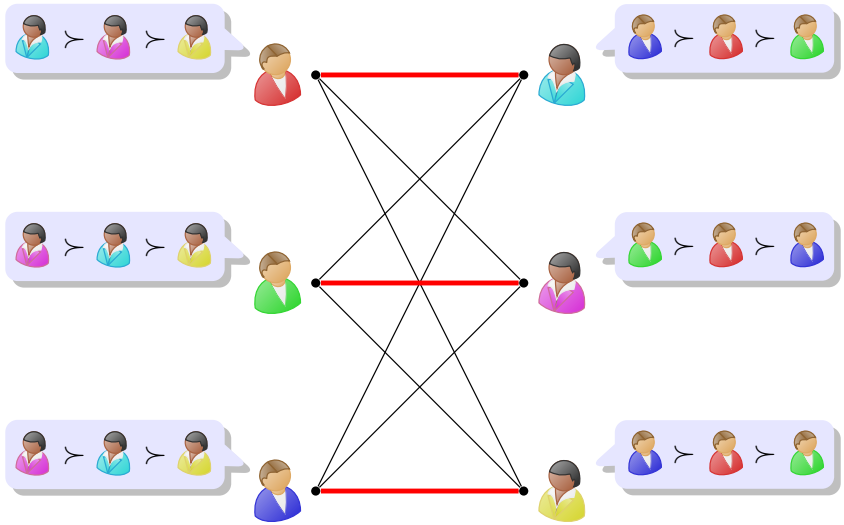


Stable matching

a matching without blocking pairs is a stable matching



stable



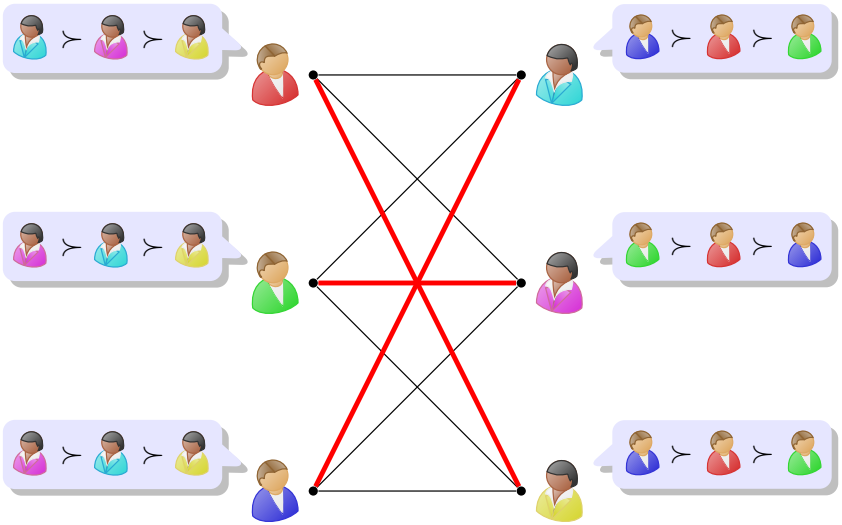
unstable

Question

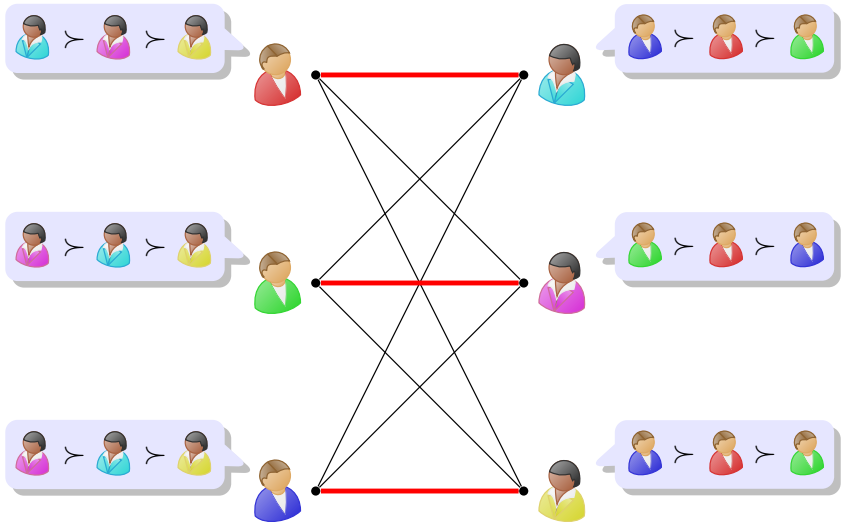
- Does a stable matching always exist?
- Efficiently computable?

Stable matching

a matching without blocking pairs is a stable matching



stable



unstable

Question

- Does a stable matching always exist?
- Efficiently computable?

→ YES! By the Gale–Shapley algorithm! [Gale and Shapley 1962]

Gale–Shapley Algorithm

```
1  Initially all m ∈ M and w ∈ W are single, and S ← ∅;
2  while ∃m ∈ M who is single do
3      w ← the most-preferred woman for m to whom he has not proposed;
4      m proposes w;
5      if w is single then
6          w accepts m (S ← S ∪ {(m, w)});
7      else if w matches with m' and m ≻_w m' then
8          w rejects m' and accepts m (S ← S \ {(m', w)} ∪ {(m, w)});
9      else
10         w rejects m;
11 Return S;
```

cf. `https://yambi.jp/stable_matching/index.html`

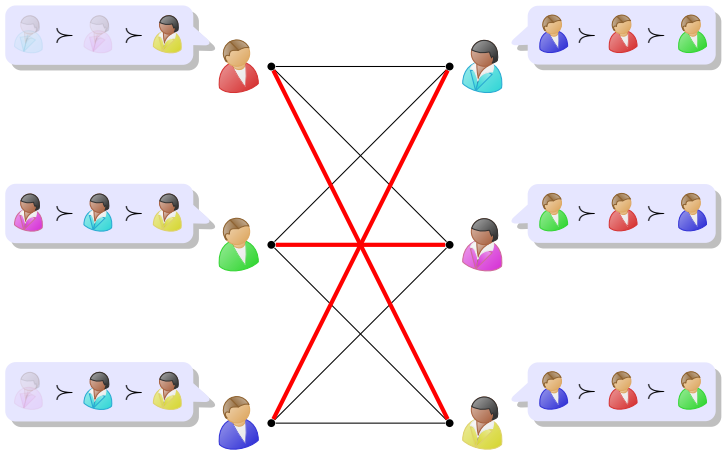
Analysis of the GS algorithm (1/2)

Theorem

The GS algorithm terminates in at most n^2 steps

Proof

- Each $m \in M$ proposes to each w at most once
- $\#$ of proposals $\leq |M| \times |W| = n^2$



Analysis of the GS algorithm (2/2)

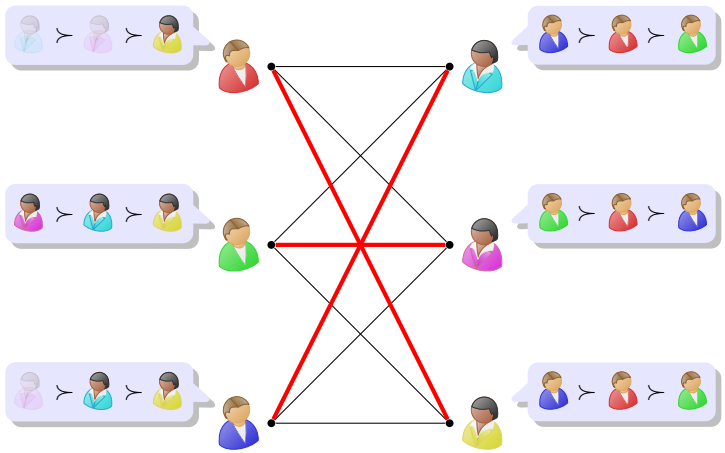
Theorem

The GS algorithm always outputs a stable matching

Proof

- Suppose to the contrary that (m, w) is a blocking pair
- Then m must propose to w in the procedure
- w's final partner must be someone more preferable than m

  → contradiction



Applications of the GS algorithm

The GS algorithm (and its extension) is used in real world

- Resident matching (US, UK, Japan,...)

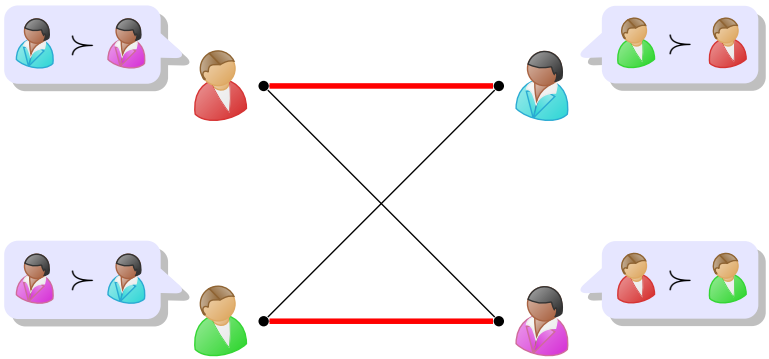
THE MATCH
NATIONAL RESIDENT MATCHING PROGRAM®

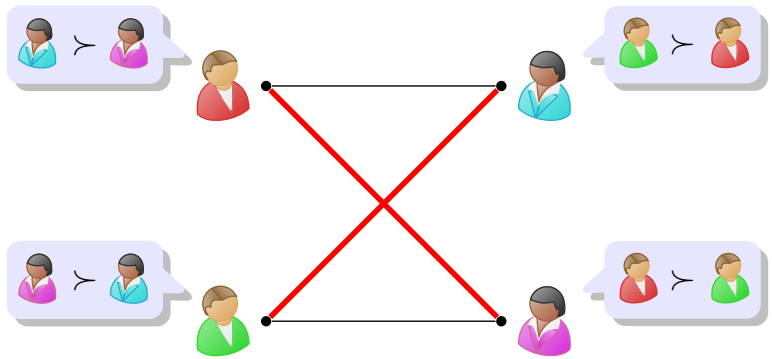
- School choice (New York, Boston,...)
- Laboratory assignments at various universities
- UTokyo 進学選択の第二段階

  `http://www.c.u-tokyo.ac.jp/zenki/news/kyoumu/firstyear/2016/1125173747.html`

Multiple stable matchings

- there may be multiple stable matchings
- any "good" structure?



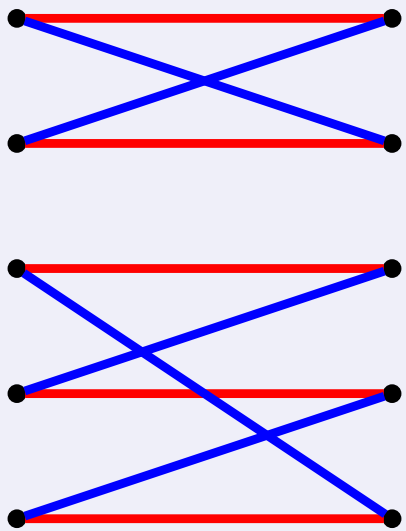


Operations

For stable matchings S_1 and S_2,

- $S_1 \vee S_2$: matching s.t. each man selects better partner in $S_1 \cup S_2$
  each woman selects worse partner in $S_1 \cup S_2$
- $S_1 \wedge S_2$: matching s.t. each man selects worse partner in $S_1 \cup S_2$
  each woman selects better partner in $S_1 \cup S_2$

$S_1 \triangle S_2$



Structure of stable matchings

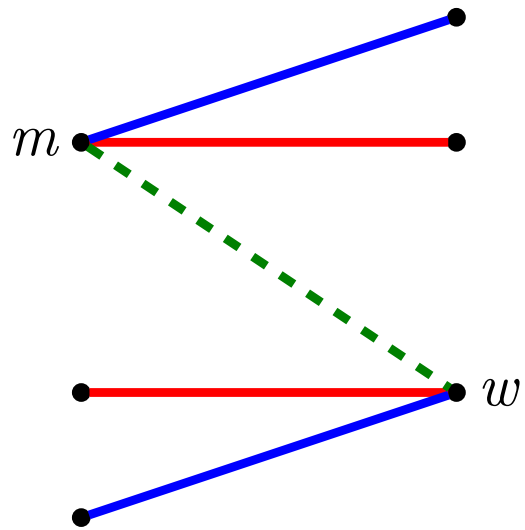
Theorem

$S_1 \vee S_2$ and $S_1 \wedge S_2$ are stable matchings

→ the stable matchings form a (distributive) lattice

→ $\exists$man-optimal stable matching and woman-optimal stable matching

Proof



Output of the GS algorithm

Theorem

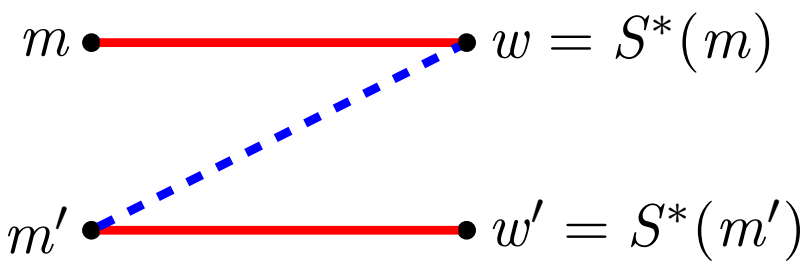
The GS algorithm outputs the man-optimal stable matching

regardless of the order in which the proposers are selected

Proof

- Let S be the output of GS and S^* be the man-optimal stable matching
- Suppose to the contrary that $S \neq S^*$, i.e., $\exists m$ was rejected by $S^*(m)$
- Let m' be the partner of $w := S^*(m)$ at the first such rejection
- Then $m' \succ_w m$ and $w \succ_{m'} w'$ → (m', w) blocks S^*

  → contradiction



Exercise

Find all the stable matchings